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Publication date:
1988

Document Version
Publisher's PDF, also known as Version of record

Risø-M-2760

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December 1988

RISØ-M-2760

A SIMPLE WAKE CALCULATION PROCEDURE

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ABSTRACT. This report deals with a first and a second order model for the description of the wake behind a wind turbine.
The models are based on the turbulent boundary equations, and closed form solutions are obtained for the width of the wake as well as for the mean wind profile in the wake, by adopting a similarity assumption and by utilizing the mixing length theory in the description of the turbulent stresses.

The models have been made accessible by being implemented into an interactive program package designed for use on a personal computer. The use of the package is described.

December 1988
Risø National Laboratory, DK-4000 Roskilde, Denmark.

ISBN 87-550-1484-4
ISSN 0418-6435

Grafisk Service, Risø 1988

Contents

1 Introduction

Dealing with two or more wind turbines placed in the immediate vicinity of each other, the interference of these become important both in relation to the structural loads and in relation to the energy production.

The energy production is primarily determined by the distribution of the mean wind velocities over the rotor plane, although the conversion from wind energy to electric energy for conventional wind turbines is a highly non-linear process. The determination of the structural loads requires knowledge of the mean wind velocities as well as the turbulence.

The present report deals with the determination of the mean wind profile in the wake behind a wind turbine as well as the width of the wake at a given distance downstream from the rotorplane.

The model is partly an application of the work by Swain [2]. In order to understand the background, to correct misprints in [2], and to introduce the boundary conditions in a different way, it is found suitable to describe the model in some detail.

Two versions of different levels of sophistication, but both based on the turbulent boundary layer equations and a similarity assumption, are described. Within this framework the first-order model only takes into account the *dominant terms* in the boundary layer equations, whereas the second-order model takes into account the full system of boundary layer equations.

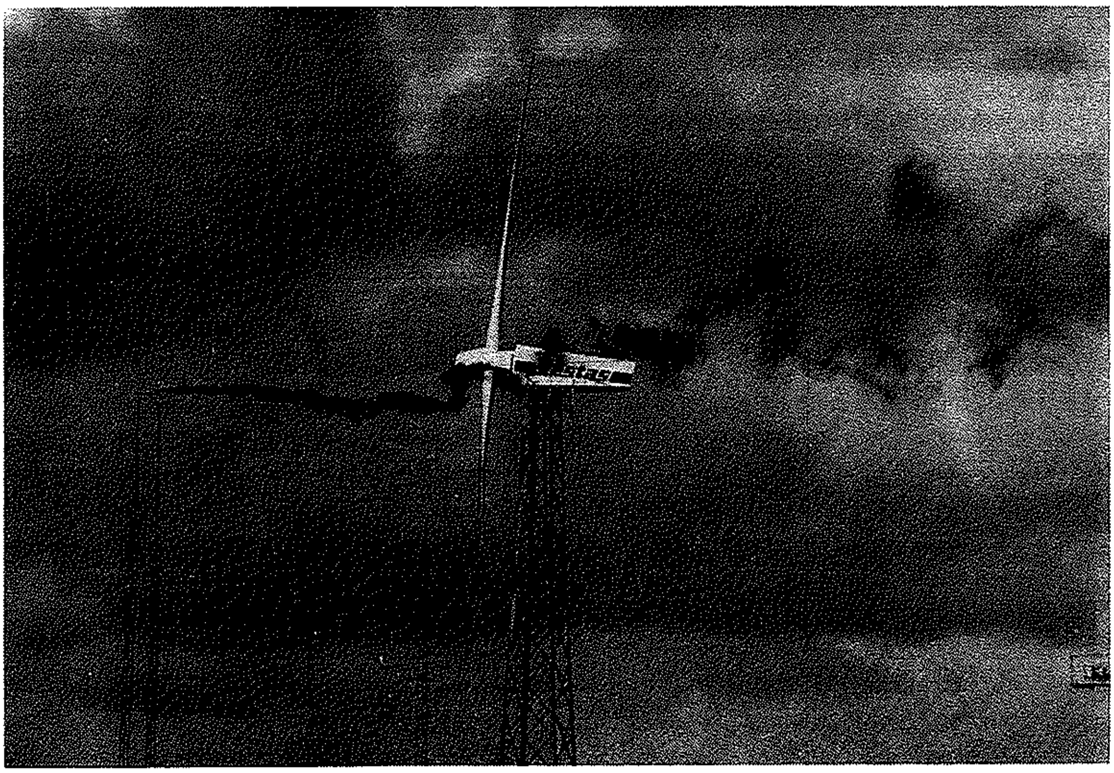
The models have been implemented in an interactive program package designed for use on a PC and written in Turbo-Pascal.

A few examples of the graphical output generated by the program are presented.

2 Fundamental assumptions

Analyses of the flow behind a wind turbine by means of a visual tracer have been conducted at the Test Station, Risø.

Those investigations indicate, as shown on the figure below [3], that the flow region behind the rotor plane is a fully turbulent region.



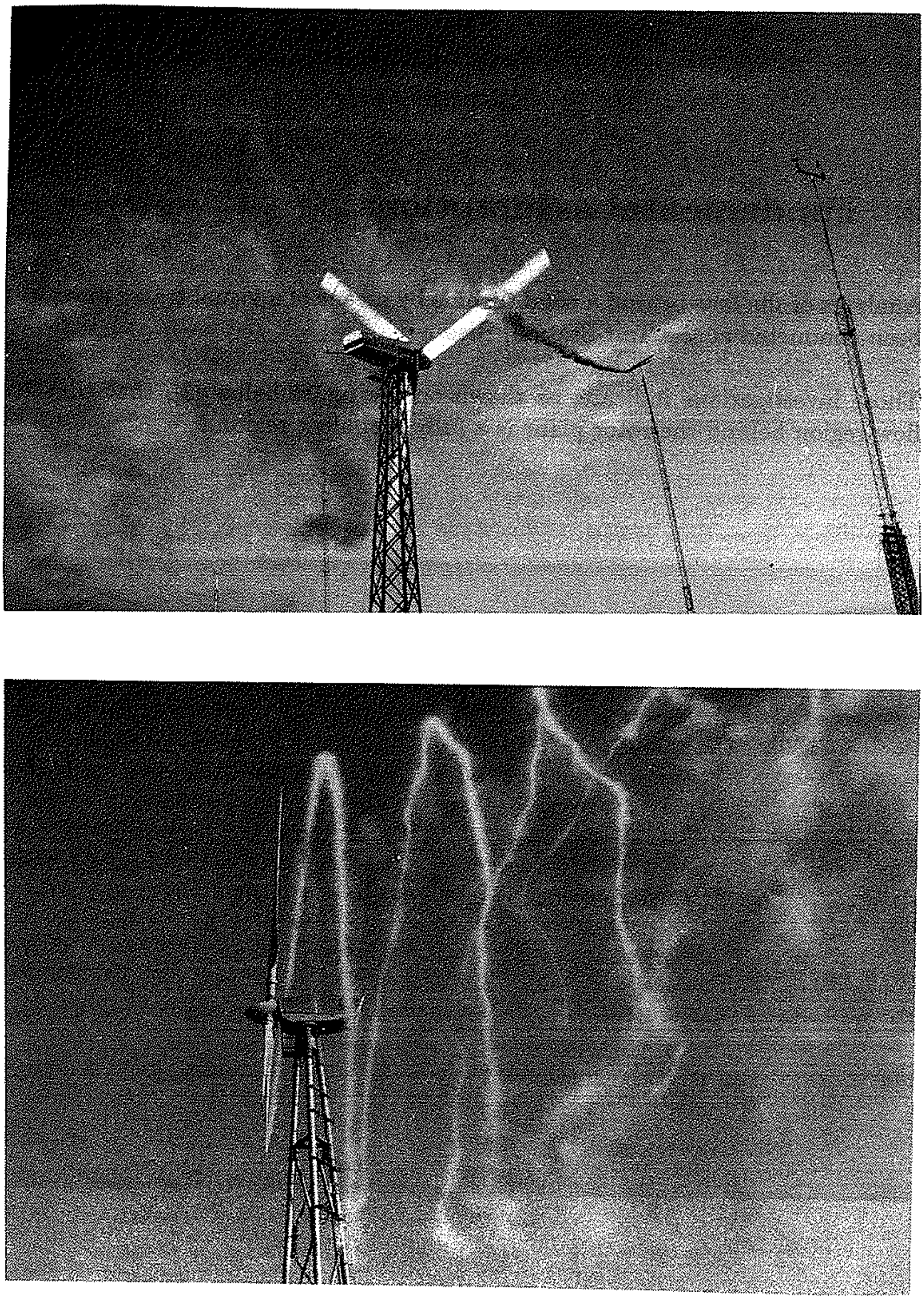


Figure 2.1: Flow visualization behind a wind turbine.

Furthermore, as stated in [1], problems involving free turbulence is often of the same nature as boundary layer problems in the sense that the size of the free turbulence region in the direction perpendicular to the mean flow is considerably smaller than the size in the mean flow direction.

The above considerations lead to the *assumption*, that the wake region behind a wind turbine can be adequately described by Prandtl's turbulent boundary layer equations.

These boundary layer equations can be considered as an asymptotic version of Navier-Stokes equations for large Reynolds numbers. As a first check of the above assumption it is therefore of interest to calculate the order of magnitude of the Reynolds number in the actual situation.

In [4] the Reynolds number, for air at standard see level conditions of density and viscosity, is given by

$$R = 69000Vl, \tag{2. 1}$$

where V and l denote a characteristic velocity and length, respectively, both given in mks units. Taking l as a typical rotordiameter of 25 m and V as 10 m/s one obtain a typical Reynolds number of $R \simeq 1.7 \cdot 10^7$, which is indeed a rather large value. For comparison it can be mentioned, that a representative Reynolds number for a propeller driven aircraft is 10^7 [4]. It is thus seen that the calculation of a typical Reynolds number supports the boundary layer assumption.

In order to achieve further simplifications of the problem the wind shear is *neglected*, and it is thus possible (inspired by the geometry of the rotor) to express the boundary layer equations in cylindrical coordinates. Finally it is *assumed* that the flow is incompressible and stationary.

3 Model equations

Let the wake be described in a cylindrical coordinate system with the axial and radial directions denoted by x and r, respectively, as shown in Fig. 3.1.

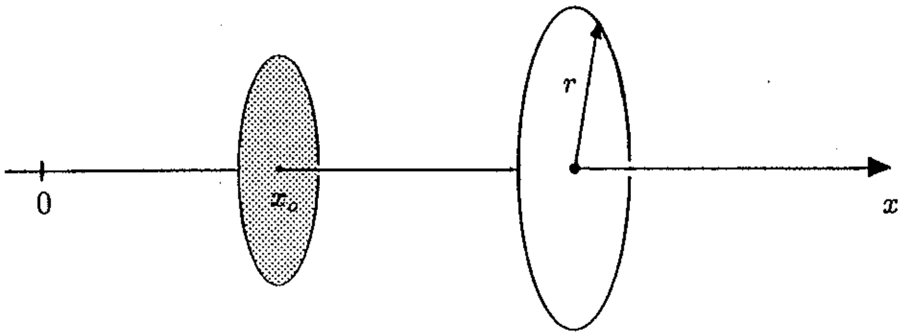


Figure 3.1: Coordinate system.

Let the position of the rotor be determined by x_o, where $x_o > 0$. Denoting the mean velocities in the wake by $U_\infty + u_x$ and u_r in the axial and radial direction, respectively, the boundary layer equations can be expressed as

$$\frac{\partial}{\partial x}((U_\infty + u_x)r) + \frac{\partial}{\partial r}(u_r r) = 0 \tag{3. 1}$$

$$(U_\infty + u_x)\frac{\partial}{\partial x}(U_\infty + u_x) + u_r\frac{\partial}{\partial r}(U_\infty + u_x) = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{1}{r}\frac{\partial}{\partial r}(r\overline{u_x^* u_r^*}) \,, \tag{3. 2}$$

where p denotes the pressure, an upper case star symbolizes a turbulent component, and an overbar is a mean value operator. The second term on the right hand side of (3.2) represents the Reynold stresses.

In [2, p.656-657] it is demonstrated that for the first and second order solutions the pressure term in (3.2) can be neglected and that the Reynold stresses, applying Prandtl's mixing length theory, can be adequately represented by $\frac{1}{r}\frac{\partial}{\partial r}\left[rl^2(\frac{\partial u_x}{\partial r})^2\right]$, where l is the mixing length, which in general is varying from point to point. Thus the above equations can be simplified to

$$\frac{\partial}{\partial x}(u_x r) \quad + \quad \frac{\partial}{\partial r}(u_r r) = 0 \tag{3. 3}$$

$$(U_\infty + u_x)\frac{\partial u_x}{\partial x} \quad + \quad u_r\frac{\partial u_x}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left[l^2 r\left(\frac{\partial u_x}{\partial r}\right)^2\right]. \tag{3. 4}$$

In what follows the momentum equation corresponding to the equation of motion (3.4) will prove useful. It can of course be obtained by integrating (3.4) but, as stated in [2], is more easily obtained directly.

Taking as a control surface an infinitly large cylindrical body surrounding the rotor and with its axis of symmetry identical with the x-axis, the following is obtained:

$$2\pi\rho\left[\int_o^{+\infty} U_\infty^2 r dr - \int_o^{+\infty}(U_\infty + u_x)^2 r dr - \int_{-\infty}^{+\infty} U_\infty u_{r\infty} r dx\right]$$

$$= \frac{1}{2}\rho c_w F U_\infty^2, \tag{3. 5}$$

where the right hand term is the resistance of the body, ρ denotes the density, $u_{r\infty}$ denotes the radial velocity at infinity, c_w is the drag coefficient and F is the rotor area.

Since, because of symmetry, $u_r = 0$ and thereby finite on the axis of symmetry ($r = 0$), the following can be stated:

$$u_{r\infty} r = 0 + \int_o^{+\infty} \frac{\partial}{\partial r}(u_r r) dr. \tag{3. 6}$$

Applying (3.3) we obtain

$$\frac{\partial}{\partial r}(u_r r) = -\frac{\partial}{\partial x}(u_x r). \tag{3. 7}$$

Using (3.6) and (3.7) the last term on the left hand side of (3.5) can be expressed as

$$\begin{aligned}
\int_{-\infty}^{+\infty} U_\infty u_{r\infty} r dx &= \int_{-\infty}^{+\infty} U_\infty \int_o^{+\infty} \frac{\partial}{\partial r}(u_r r) dr dx \\
&= -\int_{-\infty}^{+\infty} U_\infty \int_o^{+\infty} \frac{\partial}{\partial x}(u_x r) dr dx \\
&= -U_\infty \int_o^{+\infty} r \int_{-\infty}^{+\infty} \frac{\partial u_x}{\partial x} dx dr \\
&= -U_\infty \int_o^{+\infty} r [u_x]_{-\infty}^{+\infty} dr \\
&= -U_\infty \int_o^{+\infty} r u_x dr. \qquad (3.\ 8)
\end{aligned}$$

Thus finally, combining (3.8) and (3.5), we get

$$-2\pi\rho \int_o^{+\infty} (u_x^2 + U_\infty u_x) r dr = \frac{1}{2}\rho c_w F U_\infty^2 \ ,$$

or

$$\int_o^{+\infty} (u_x^2 + U_\infty u_x) r dr = -\frac{1}{4\pi} c_w F U_\infty^2. \qquad (3.\ 9)$$

4 First order wake model

The first-order system is obtained by neglecting certain terms in the governing equations and is thus only an approximate solution. However, the description given by this solution might be satisfactory and, if not, it is definitely a necessary step towards obtaining the second-order solution within the present framework.

4.1 Simplified equations

The basic idea, guided by Prandtl [5, p.65], is here to *assume* that for large Reynolds' numbers the velocities in sections perpendicular to the direction of flow are mechanically and geometrically similar. To quantify, it is *assumed* that the size of the boundary of the turbulent wake is proportional to x^n, and that the axial velocity can be expressed by

$$u_x \equiv x^m g\left(\frac{r}{k_1 x^n}\right) \equiv x^m f(\frac{r}{x^n}), \tag{4.1. 1}$$

where k_1 is a proportionality constant and the exponents m and n have to be determined. Introducing the new variable η as

$$\eta \equiv \frac{r}{x^n} \tag{4.1. 2}$$

in the equation (3.9) and neglecting, as a first approximation the term u_x^2, we obtain

$$\int_o^{+\infty} U_\infty x^{m+2n} f(\eta)\eta d\eta = -\frac{1}{4\pi} c_w F U_\infty^2. \tag{4.1. 3}$$

As the right hand side is a constant the equation can only be satisfied, provided that $U_\infty x^{m+2n}$ is a constant which again means that

$$m + 2n = 0 \ ,$$

or

$$m = -2n. \tag{4.1. 4}$$

Equation (4.1.1) can thus be expressed as

$$u_x = x^{-2n} f(\eta). \tag{4.1. 5}$$

A similarity assumption, Prandtl [5, p.66], shows that

$$l \frac{\partial}{\partial r}(U_\infty + u_x) \sim \frac{Dr_o}{Dt} \ , \tag{4.1. 6}$$

and that

$$l \sim r_o = k_1 x^n, \tag{4.1. 7}$$

where r_o denotes the width of the wake, $\sim$ denotes proportionallity, and $\frac{D(\cdot)}{Dt}$ denote absolute differentiation. From (4.1.6) the following results:

$$\begin{aligned} l\frac{\partial u_x}{\partial r} &\sim \frac{\partial r_o}{\partial t} + \frac{\partial r_o}{\partial x}\frac{dx}{dt} + \frac{\partial r_o}{\partial r}\frac{dr}{dt} \\ &= \frac{\partial r_o}{\partial x}(U_\infty + u_x) \simeq \frac{\partial r_o}{\partial x} U_\infty, \end{aligned} \tag{4.1. 8}$$

where it has been utilized that the flow is assumed stationary, that $\frac{\partial r_o}{\partial r} = 0$, and u_x *is considered small* compared with U_∞.

Introducing

$$l \equiv k_2 x^n \tag{4.1. 9}$$

in (4.1.8) together with (4.1.7) and (4.1.5) yields:

$$k_2 x^n \frac{\partial}{\partial r}(x^{-2n} f(\eta)) \equiv k_3 U_\infty \frac{\partial}{\partial x}(k_1 x^n) \ ,$$

or

$$k_2 x^n x^{-2n} x^{-n} f'(\eta) = k_3 U_\infty k_1 n x^{n-1} \ ,$$

or

$$[k_2 f'(\eta)] x^{-2n} = [k_1 k_3 n U_\infty] x^{n-1} \ ,$$

where k_2 and k_3 are proportionality constants and $(\cdot)'$ denote differentiation with respect to η.

The above equation has to be true for all x, whereby the following relation is derived

$$-2n = n - 1 \ ,$$

or

$$n = \frac{1}{3} \ . \tag{4.1. 10}$$

Thus we finally get the following expression for the velocity component u_x:

$$u_x = x^{-\frac{2}{3}} f(\frac{r}{x^{\frac{1}{3}}}) = x^{-\frac{2}{3}} f(\eta) \ . \tag{4.1. 11}$$

The next step is to determine the order of size of the various terms in the equation of motion (3.4).

We obtain:

$$\begin{aligned} U_\infty \frac{\partial u_x}{\partial x} &= U_\infty \left[-\frac{2}{3} x^{-\frac{5}{3}} f(\eta) - \frac{1}{3} x^{-\frac{2}{3}} x^{-\frac{4}{3}} r f'(\eta) \right] \\ &= U_\infty \left[-\frac{2}{3} x^{-\frac{5}{3}} f(\eta) - \frac{1}{3} x^{-\frac{5}{3}} \eta f'(\eta) \right], \qquad (4.1.\ 12) \\ u_x \frac{\partial u_x}{\partial x} &= x^{-\frac{2}{3}} f(\eta) \left[-\frac{2}{3} x^{-\frac{5}{3}} f(\eta) - \frac{1}{3} x^{-\frac{5}{3}} \eta f'(\eta) \right] \\ &= f(\eta) \left[-\frac{2}{3} x^{-\frac{7}{3}} f(\eta) - \frac{1}{3} x^{-\frac{7}{3}} \eta f'(\eta) \right], \qquad (4.1.\ 13) \end{aligned}$$

and

$$\begin{aligned} u_r \frac{\partial u_x}{\partial r} &= u_r x^{-\frac{2}{3}} x^{-\frac{1}{3}} f'(\eta) \\ &= u_r x^{-\frac{3}{3}} f'(\eta). \qquad (4.1.\ 14) \end{aligned}$$

Now, with the aid of (3.3), expressing u_r as

$$\begin{aligned} u_r &= -\frac{1}{r} \int_o^r \frac{\partial}{\partial x} (u_x r) dr \\ &= \frac{1}{\eta} \int_o^\eta \left[\frac{2}{3} x^{-\frac{5}{3}} f(\eta) + \frac{1}{3} x^{-\frac{5}{3}} \eta f'(\eta) \right] x^{\frac{1}{3}} \eta d\eta \\ &= \frac{1}{3\eta} x^{-\frac{4}{3}} \int_o^\eta \frac{\partial}{\partial \eta} [\eta^2 f(\eta)] d\eta \\ &= \frac{\eta}{3} x^{-\frac{4}{3}} f(\eta) \end{aligned}$$

it is seen, that (4.1.14) can be formulated as

$$u_r \frac{\partial u_x}{\partial r} = \frac{\eta}{3} x^{-\frac{7}{3}} f(\eta) f'(\eta). \tag{4.1. 15}$$

Finally the last term yields:

$$
\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial r}\left[l^2 r\left(\frac{\partial u_x}{\partial r}\right)^2\right] &= \frac{1}{r}\frac{\partial}{\partial r}\left[k_2^2 x^{\frac{2}{3}} r\left(x^{-\frac{2}{3}}x^{-\frac{1}{3}}f'(\eta)\right)^2\right] \\
&= \frac{1}{r}k_2^2 x^{-\frac{4}{3}}(f'(\eta))^2 + 2k_2^2 x^{-\frac{4}{3}}f''(\eta)x^{-\frac{1}{3}} \\
&= \frac{1}{\eta}k_2^2 x^{-\frac{5}{3}}(f'(\eta))^2 + 2k_2^2 x^{-\frac{5}{3}}f''(\eta). \qquad (4.1.\ 16)
\end{aligned}
$$

One of our assumptions is that $U_\infty >> u_x$, and consequently we must be at some distance from the rotor. From (4.1.12), (4.1.13), (4.1.15) and (4.1.16) it is seen that at this distance the terms behave like $x^{-\frac{5}{3}}, x^{-\frac{7}{3}}, x^{-\frac{7}{3}}$ and $x^{-\frac{5}{3}}$, respectively.

Now, in the first order approximation, we only take into account the terms of the order of magnitude $x^{-\frac{5}{3}}$. Thus the equation of motion (3.4) reduces to

$$
U_\infty \frac{\partial u_x}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left[l^2 r\left(\frac{\partial u_x}{\partial r}\right)^2\right], \qquad (4.1.\ 17)
$$

which together with the continuity equation (3.3) form the system of equations for the first order problem.

4.2 Solution of first order equations

In order to obtain the results in non-dimensional variables, the χ_1 function is introduced as

$$
u_x = x^{-\frac{2}{3}} f(rx^{-\frac{1}{3}}) \equiv U_\infty (c_w F x^{-2})^{\frac{1}{3}} \chi_1(r(c_w F x)^{-\frac{1}{3}}). \qquad (4.2.\ 1)
$$

Furthermore, introducing the variable ζ as

$$\zeta \equiv r(c_w F x)^{-\frac{1}{3}}, \tag{4.2. 2}$$

the above relation reads:

$$u_x = U_\infty (c_w F x^{-2})^{-\frac{1}{3}} \chi_1(\zeta). \tag{4.2. 3}$$

In the same way we express the mixing length, given by (4.1.9-10), in the non-dimensional form

$$l = k_2 x^{\frac{1}{3}} \equiv c_1 (c_w F x)^{\frac{1}{3}}. \tag{4.2. 4}$$

c_1 is a constant for the actual problem and has to be determined by measurements. In the above definition of c_1 the effect of drag from the rotor has in some way been separated (through the parameter $c_w F$), and consequently c_1 is expected to be rather insensitive to the form and the size of the rotor.

We are now in a position to solve the first order problem. By introducing the above relations (4.2.3-4) in the first order equation (4.1.17), the following is obtained:

$$U_\infty \frac{\partial}{\partial x}\left[U_\infty (c_w F x^{-2})^{\frac{1}{3}} \chi_1(\zeta)\right]$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left[c_1^2 (c_w F x)^{\frac{2}{3}} r \left\{\frac{\partial}{\partial r}[U_\infty (c_w F x^{-2})^{\frac{1}{3}} \chi_1(\zeta)]\right\}^2\right] \tag{4.2. 5}$$

or, as shown in appendix A

$$
\begin{aligned}
c_1^2 \frac{d}{d\zeta}\left[\zeta\left\{\frac{d}{d\zeta}[\chi_1(\zeta)]\right\}^2\right] &= -\frac{2}{3}\zeta\chi_1(\zeta) - \frac{1}{3}\zeta^2\frac{d}{d\zeta}[\chi_1(\zeta)] \\
&= -\frac{1}{3}\frac{d}{d\zeta}[\zeta^2\chi_1(\zeta)]. \qquad (4.2.\ 6)
\end{aligned}
$$

Integrating (4.2.6) we get

$$
c_1^2\zeta\left\{\frac{d}{d\zeta}[\chi_1(\zeta)]\right\}^2 = -\frac{1}{3}\zeta^2\chi_1(\zeta) + IC_1, \qquad (4.2.\ 7)
$$

where IC_1 denotes an integration constant. It is obvious, that $\chi_1(\zeta)$ and $\frac{\partial}{\partial\zeta}[\chi_1(\zeta)]$ must be finite at the axis of symmetry where $\zeta = 0$ (cf. (4.2.3)). Thus the above relation (4.2.7) imply, that $IC_1 = 0$, whereby the following relation is obtained:

$$
3c_1^2\zeta\left\{\frac{d}{d\zeta}[\chi_1(\zeta)]\right\}^2 + \zeta^2\chi_1(\zeta) = 0. \qquad (4.2.\ 8)
$$

Finally the variable defined by

$$
\xi \equiv (3c_1^2)^{-\frac{1}{3}}\zeta, \qquad (4.2.\ 9)
$$

and the function defined by

$$
\chi_{11}(\xi) \equiv \chi_1(\zeta) \qquad (4.2.\ 10)
$$

is introduced. Equation (4.2.8) is thus reduced to

$$3c_1^2\xi(3c_1^2)^{\frac{1}{3}}\left\{\frac{d}{d\xi}[\chi_{11}(\xi)]\frac{d\xi}{d\zeta}\right\}^2+\xi^2(3c_1^2)^{\frac{2}{3}}\chi_{11}(\xi)=0,$$

which means that

$$\frac{d}{d\xi}[\chi_1(\xi)]=^{+}_{-}(-\xi\chi_{11}(\xi))^{\frac{1}{2}} \ ,$$

or

$$\frac{d}{d\xi}[\chi_{11}(\xi)]\{\chi_{11}(\xi)\}^{-\frac{1}{2}}=\frac{d}{d\xi}\left[2\{\chi_{11}(\xi)\}^{\frac{1}{2}}\right]=^{+}_{-}i\xi^{\frac{1}{2}}. \qquad (4.2.\ 11)$$

Integrating the above equation we obtain

$$^{-}_{+}2\{\chi_{11}(\xi)\}^{\frac{1}{2}}=i\frac{2}{3}\xi^{\frac{3}{2}}-2iIC_2 \ ,$$

or

$$\chi_{11}(\xi)=-\{\frac{1}{3}\xi^{\frac{3}{2}}-IC_2\}^2 \ , \qquad (4.2.\ 12)$$

where IC_2 is an integration constant, which has to be determined from the boundary conditions.

The boundary conditions are:

(b1) $u_x=0$ on the boundary of the wake represented by $\xi=\xi_o$.

(b2) $\int_o^{+\infty}U_\infty u_x r\,dr=-\frac{1}{4\pi}c_w F U_\infty^2$. This equation is obtained from (3.9) utilizing, that in the first order approximation, we only retain the largest terms in the equation of motion.

As $u_x=0$ imply that $\chi_{11}(\xi)=0$, we have immediately from boundary condition (b1) that

$$IC_2 = \frac{1}{3}\xi_o^{\frac{3}{2}} \ . \tag{4.2. 13}$$

From the boundary condition (b2) we find, utilizing (4.2.3) and (4.2.9-10):

$$\begin{aligned} -\frac{1}{4\pi}c_w F U_\infty^2 = \int_o^{+\infty} U_\infty u_x r dr &= \int_o^{+\infty} U_\infty^2 (c_w F x^{-2})^{\frac{1}{3}} \zeta (c_w F x)^{\frac{2}{3}} \chi_1(\zeta) d\zeta \\ &= \int_o^{+\infty} U_\infty^2 c_w F (3c_1^2)^{\frac{2}{3}} \xi \chi_{11}(\xi) d\xi \\ &= \int_o^{\xi_o} U_\infty^2 c_w F (3c_1^2)^{\frac{2}{3}} \xi \chi_{11}(\xi) d\xi \ , \end{aligned}$$

as $\chi_{11}(\xi) = 0$ outside the boundary of the wake $\xi = \xi_o$.

Introducing (4.2.11) and (4.2.12) in the above we get:

$$-\frac{1}{4\pi} = -\int_o^{\xi_o} (3c_1^2)^{\frac{2}{3}} \xi \left\{ \frac{1}{3}\xi^{\frac{3}{2}} - \frac{1}{3}\xi_o^{\frac{3}{2}} \right\}^2 d\xi \ ,$$

or

$$\begin{aligned} \frac{1}{4\pi}(3c_1^2)^{-\frac{2}{3}} &= \left[\frac{1}{45}\xi^5 - \frac{4}{63}\xi_o^{\frac{3}{2}}\xi^{\frac{7}{2}} + \frac{1}{18}\xi_o^3\xi^2 \right]_o^{\xi_o} \\ &= \frac{1}{70}\xi_o^5 \ . \end{aligned} \tag{4.2. 14}$$

Thus from (4.2.11-13) the following expressions for the boundary of the wake and for the velocity deficit can be obtained:

$$\begin{aligned} \xi_o &= \left(\frac{35}{2\pi}\right)^{\frac{1}{5}} (3c_1^2)^{-\frac{2}{15}} \ , \\ u_x &= -U_\infty (c_w F x^{-2})^{\frac{1}{3}} \left\{ \frac{1}{3}\xi^{\frac{3}{2}} - \frac{1}{3}\left(\frac{35}{2\pi}\right)^{\frac{3}{10}} (3c_1^2)^{-\frac{1}{5}} \right\}^2 \ , \end{aligned}$$

or, expressed in the basic variables x and r

$$
\begin{aligned}
r_o &= \left(\frac{35}{2\pi}\right)^{\frac{1}{5}} (3c_1^2)^{\frac{1}{5}} (c_w F x)^{\frac{1}{3}}, && (4.2.\ 15)\\
u_x &= -\frac{U_\infty}{9}\left(c_w F x^{-2}\right)^{\frac{1}{3}} \Bigg\{ r^{\frac{3}{2}} (3c_1^2 c_w F x)^{-\frac{1}{2}} \\
&\quad - \left(\frac{35}{2\pi}\right)^{\frac{3}{10}} (3c_1^2)^{-\frac{1}{5}} \Bigg\}^2 , && (4.2.\ 16)
\end{aligned}
$$

where r_o denotes the radius of the turbulente wake.

Finally, the radial velocity u_r can be obtained from the equation of continuity (3.3):

$$
\begin{aligned}
u_r &= \frac{1}{r}\left[0 + \int_o^r \frac{\partial}{\partial r}(u_r r) dr\right] \\
&= -\frac{1}{r}\int_o^r \frac{\partial}{\partial x}(u_x r) dr \\
&= -\frac{1}{r}\int_o^r r \frac{\partial}{\partial x}(u_x) dr \\
&= -\frac{1}{\xi}(3c_1^2 c_w F x)^{\frac{1}{3}} \int_o^\xi \frac{\partial}{\partial x}\left[U_\infty (c_w F x^{-2})^{\frac{1}{3}} \chi_{11}(\xi)\right] r (3c_1^2 c_w F x)^{-\frac{1}{3}} d\xi \\
&= -\frac{1}{\xi} U_\infty (3c_1^2 c_w^2 F^2 x)^{\frac{1}{3}} \int_o^\xi \frac{\partial}{\partial x}\left[x^{-\frac{2}{3}} \chi_{11}(\xi)\right] \xi d\xi \\
&= -\frac{1}{\xi} U_\infty (3c_1^2 c_w^2 F^2 x)^{\frac{1}{3}} \int_o^\xi \left(-\frac{2}{3} x^{-\frac{5}{3}} \chi_{11}(\xi) + x^{-\frac{2}{3}} \frac{d}{d\xi}\left[\chi_{11}(\xi)\right] \frac{\partial \xi}{\partial x}\right) \xi d\xi \\
&= \frac{1}{\xi} U_\infty (3c_1^2 c_w^2 F^2 x)^{\frac{1}{3}} x^{-\frac{5}{3}} \int_o^\xi \left(\frac{2}{3}\xi \chi_{11}(\xi) + \frac{1}{3}\xi^2 \frac{d}{d\xi}\left[\chi_{11}(\xi)\right]\right) d\xi \\
&= \frac{1}{\xi} U_\infty (3c_1^2 c_w^2 F^2)^{\frac{1}{3}} x^{-\frac{4}{3}} \int_o^\xi \frac{d}{d\xi}\left[\frac{1}{3}\xi^2 \chi_{11}(\xi)\right] d\xi \\
&= \frac{U_\infty}{3}(3c_1^2 c_w^2 F^2)^{\frac{1}{3}} x^{-\frac{4}{3}} \xi \chi_{11}(\xi) , && (4.2.\ 17)
\end{aligned}
$$

as $\chi_{11}(\xi)$ is finite for $\xi = 0$. Transforming to the basic variables x and r we get:

$$u_r = \frac{U_\infty}{3}(c_w F)^{\frac{1}{3}} x^{-\frac{5}{3}} r \left\{ r^{\frac{3}{2}}(3c_1^2 c_w F x)^{-\frac{1}{2}} - \left(\frac{35}{2\pi}\right)^{\frac{3}{10}} (3c_1^2)^{-\frac{1}{5}} \right\}^2 . \tag{4.2. 18}$$

Summing up, it is seen that the boundary of the wake is given by (4.2.15), and that the velocity field inside the wake can be determined from (4.2.16) and (4.2.18).

4.3 Calibration of the model

The expressions obtained in section 4.2 contain two unknown constants, which have to be determined before an explicit calculation can be performed. These are the constant c_1, related to the mixing length, and the position of the rotor, x_o, relatively to the applied coordinate system.

In order to calculate x_o and c_1 two conditions to be satisfied are required. First we *assume* that the width of the wake at the position of the wind turbine, x_o, equals the diameter of the rotor. Secondly we need a measurement of the axial velocity at the axis of symmetry at some position $x_o + \Delta x$ downstream the wake.

A measurement point at the axis of symmetry is chosen in order to simplify the expressions, and because the influence from the form of the obstacle at moderate Δx probably is less at the axis than outside the axis.

The conditions to be satisfied then read

$$D = 2\alpha_1 c_1^{\frac{2}{5}} x_o^{\frac{1}{3}} , \tag{4.3. 1}$$

and

$$U_m = U_\infty \left\{ 1 - \beta_1 c_1^{-\frac{4}{5}} (x_o + \Delta x)^{-\frac{2}{3}} \right\} , \tag{4.3. 2}$$

where

$$\alpha_1 = \left(\frac{105}{2\pi}\right)^{\frac{1}{5}} (c_w F)^{\frac{1}{3}}, \tag{4.3. 3}$$

$$\beta_1 = \frac{1}{9} 3^{-\frac{2}{5}} \left(\frac{35}{2\pi}\right)^{\frac{3}{5}} (c_w F)^{\frac{1}{3}}, \tag{4.3. 4}$$

and U_m denote the measured axial velocity. D is the diameter of the rotor. The above expressions are obtained directly from (4.2.15) and (4.2.16).

From (4.3.1) and (4.3.2) it is seen, that

$$c_1^{\frac{4}{5}} x_o^{\frac{2}{3}} = \left(\frac{D}{2\alpha_1}\right)^2 ,$$

and

$$c_1^{\frac{4}{5}} (x_o + \Delta x)^{\frac{2}{3}} = \left(\frac{\beta_1}{U_\infty - U_m}\right)^{\frac{1}{2}} .$$

From the above the following expressions for x_o and c_1 are easily obtained:

$$x_o = \left\{ \left(\frac{D}{2\alpha_1}\right)^{-3} \left(\frac{\beta_1}{U_\infty - U_m}\right)^{\frac{3}{4}} - 1 \right\}^{-1} \Delta x , \tag{4.3. 5}$$

and

$$c_1 = \left(\frac{D}{2\alpha_1}\right)^{\frac{5}{2}} x_o^{-\frac{5}{6}}. \tag{4.3. 6}$$

Computing the drag coefficient for the rotor c_w by an aerodynamic model, or alternatively estimating it by the procedure given in Appendix D, the system consisting of equations (4.3.5-6), (4.2.15-16) and (4.2.18) offer a first order estimate of the wake diameter as well as the wake velocity field.

5 Second order wake model

The second order system provide, like the first order system, only an approximate solution.

However, by adding a term to the axial velocity field from the first order problem, it is possible to take into account terms in the equation of motion of the order of magnitude $x^{-\frac{7}{3}}$, whereas the first order model only retains terms of the order of magnitude $x^{-\frac{5}{3}}$.

5.1 Simplified equations

As in section 4 we adopt Prandtl's idea, and *assume* a similarity solution, where the boundary of the turbulent wake is proportianal to x^n. If the first order axial velocity field represented by (4.1.11) is inserted in the impulse equation (3.9) without, as in the first order approximation, neglecting the term u_x^2, the following is obtained

$$\int_o^\infty \left\{U_\infty x^{-\frac{2}{3}} f(\eta) + x^{-\frac{4}{3}} f^2(\eta)\right\} x^{\frac{2}{3}} \eta d\eta = -\frac{1}{4\pi} c_w F U_\infty^2. \qquad (5.1.\ 1)$$

As the exponent of x on the left hand side obviously do not vanish, this equation can not be satisfied. However, guided by the above expression, it is seen that if u_x is *assumed* to be of the form

$$u_x = x^{-\frac{2}{3}} f_1(\eta) + x^{-\frac{4}{3}} f_2(\eta), \qquad (5.1.\ 2)$$

then (5.1.1) can be written as

$$\begin{aligned} \int_o^\infty \{U_\infty x^{-\frac{2}{3}} f_1(\eta) + U_\infty x^{-\frac{4}{3}} f_2(\eta) \quad &+ \quad x^{-\frac{4}{3}} f_1^2(\eta) + o(x^{-\frac{4}{3}})\} x^{\frac{2}{3}} \eta d\eta \\ &= \quad -\frac{1}{4\pi} c_w F U_\infty^2, \qquad (5.1.\ 3) \end{aligned}$$

and it is thus possible, by suitable choice of f_2, to take into account the first approximation of the term u_x^2. Continuing adding terms in the expression for the axial velocity, gradually better approximations of the term u_x^2 can be obtained.

The second order wake model is obtained by adopting an axial velocity of the form (5.1.2) to the full equation of motion (3.4).

The reason why the full system has to be adopted is the following. In section 4.1 it was shown, that the leading terms in the first order approximation behaved like $x^{-\frac{5}{3}}$ and that the neglected terms behaved like $x^{-\frac{7}{3}}$. In the second order approximation the leading terms containing $f_2(\eta)$ will thus behave like $x^{-\frac{5}{3}-\frac{2}{3}} = x^{-\frac{7}{3}}$ which corresponds to the order of magnitude of the terms neglected in the first order approximation.

Those can therefore not be neglected in the second order problem.

The next natural step is thus to retain terms of order $x^{-\frac{7}{3}}$ in (3.4) with the above assumed behaviour of u_x with x.

In analogy with section 4 we want to express the results in non-dimensional variables, and the axial velocity is therefore introduced as

$$u_x = U_\infty \left\{ (c_w F x^{-2})^{\frac{1}{3}} \chi_1(\zeta) + \left(c_w F x^{-2}\right)^{\frac{2}{3}} \chi_2(\zeta) \right\}, \tag{5.1. 4}$$

where ζ is defined by (4.2.2). In [2] the expression for the mixing length l is changed to include an additional term in $x^{-\frac{1}{3}}$, in order to obtain a new term on the right hand side of (3.4) of order $x^{-\frac{7}{3}}$ by combining this additional part of l with the first order approximation of u_x.

However, the only reason for changing the form of l is that they *want* a modified expression for boundary of the wake, as the right hand side contain a number of terms or order $x^{-\frac{7}{3}}$ even without changing l (cf. Appendix C). This correction of the boundary is controlled by the *assumed* form of l.

Furthermore, a result of the change of l is that it is impossible to obtain a closed form solution of the system because of the boundary conditions. An asymptotic expression is presented in [2], but for these very large values of x, the correction to l and thereby to the boundary of the wake is without importance!

Instead we adopt the form of l given by (4.2.4). This is also consistent with Prandtl's assumption (4.1.7), as we have already *assumed*, by expressing u_x as a function of $x^{-\frac{1}{3}}$, that the boundary of the wake behaves like $x^{\frac{1}{3}}$.

Substituting (4.2.4) and (5.1.4) into the equation of motion (3.4), and retaining terms of order $x^{-\frac{7}{3}}$, we obtain (cf. Appendix B, (B.6)):

$$\begin{aligned} 4\chi_2(\zeta) \quad &+ \quad \frac{d}{d\zeta}[\chi_2(\zeta)]\left\{\zeta + 6c_1^2\left(\frac{1}{\zeta}\frac{d}{d\zeta}[\chi_1(\zeta)] + \frac{d^2}{d\zeta^2}[\chi_1(\zeta)]\right)\right\} \\ &+ \quad 6c_1^2\frac{d}{d\zeta}[\chi_1(\zeta)]\frac{d^2}{d\zeta^2}[\chi_2(\zeta)] = -2\chi_1(\zeta)^2. \end{aligned} \tag{5.1. 5}$$

The above equation together with the continuity equation (3.3), or the modified version (B.1), is the system of equations describing the second order problem.

5.2 Solution of the second order problem

In order to solve the second order problem we introduce the independent variable z and the dependent variable $\chi_{22}(z)$ given by [2]:

$$z \equiv \frac{\zeta^{\frac{3}{2}}}{(3c_1^2)^{\frac{1}{2}}\xi_o^{\frac{3}{2}}}\,, \tag{5.2. 1}$$

and

$$\chi_{22}(z) \equiv \chi_2(\zeta) \tag{5.2. 2}$$

into equation (5.1.5). Furthermore, introducing the solution of the first order problem, we obtain the following expression:

$$\begin{aligned} 3z(z-1)\frac{d^2}{dz^2}[\chi_{22}(z)] &+ 3(z-1)\frac{d}{dz}[\chi_{22}(z)] \\ -8\chi_{22}(z) &= \frac{4}{81}\xi_o^6(z-1)^4 \,. \end{aligned} \tag{5.2. 3}$$

The manipulations leading to the above equation are shown in Appendix C.

From the solution of the first order problem we know, that the boundary of the wake is given by

$$\xi = (3c_1^2)^{-\frac{1}{3}}\zeta = \xi_o \,,$$

or

$$\zeta = (3c_1^2)^{\frac{1}{3}}\xi_o \,.$$

By comparing the above with (5.2.1) it is seen, that the introduction of the z-variable is a normalization which transforms the radial coordinate of the wake to the interval [0;1].

Equation (5.2.3) is a linear second order inhomogeneous differential equation and the solution is thus obtained as the complete integral of the corresponding homogeneous equation together with a particular integral satisfying the inhomogeneous equation.

The homogeneous equation is of hypergeometric type with regular singularities at 0, 1 and ∞. As indicated above we seek the solution around the singularity 0. The standard form of the hypergeometric equation is [6]:

$$\begin{aligned} z(1-z)\frac{d^2}{dz^2}[\chi_{22}(z)] &+ \{c-(a+b+1)z\}\frac{d}{dz}[\chi_{22}(z)] \\ &- ab\chi_{22}(z) = 0 \,. \end{aligned} \tag{5.2. 4}$$

By indentifying coefficients of (5.2.3) and (5.2.4) we obtain,

$$c = 1,$$
$$a + b + 1 = 1,$$
$$-ab = \frac{8}{3},$$

or

$$c = 1,$$
$$a = \frac{\mp 2\sqrt{2}}{3},$$
$$b = \frac{\pm 2\sqrt{2}}{3}.$$

It is seen that two sets of parameters satisfy the conditions. This was to be expected as the hypergeometric function is symmetric in its first two parameters. We *choose* the set

$$(a, b, c) = \left(\frac{2\sqrt{2}}{3}, \frac{-2\sqrt{2}}{3}, 1\right) . \tag{5.2. 5}$$

As c=1, the complete solution of the homogeneous equation in the neighbourhood of 0 is given by [6, p.563-564]:

$$\begin{aligned}
\chi_{22h}(z) &= C_{h1} F\left(\frac{2\sqrt{2}}{3}, -\frac{2\sqrt{2}}{3}, 1; z\right) \\
&+ C_{h2}\left\{F\left(\frac{2\sqrt{2}}{3}, -\frac{2\sqrt{2}}{3}, 1; z\right) \ln z\right. \\
&+ \sum_{n=1}^{\infty} \frac{(\frac{2\sqrt{2}}{3})_n(\frac{-2\sqrt{2}}{3})_n}{(n!)^2} z^n[\psi(\frac{2\sqrt{2}}{3}+n) - \psi(\frac{2\sqrt{2}}{3}) + \psi(-\frac{2\sqrt{2}}{3}+n) \\
&- \left.\psi(-\frac{2\sqrt{2}}{3}) - 2\psi(n+1) + 2\psi(1)]\right\} ,
\end{aligned} \tag{5.2. 6}$$

where $(\cdot)_n$ denotes Pochhammer's symbol and $\psi(\cdot)$ is the logaritmic derivative of the gamma function. C_{h1} and C_{h2} are arbitrary constants.

It is observed that the right hand term of (5.2.3) is a fourth order polynomial in z. By substitution of the polynomial

$$\chi_{22i}(z) = \sum_{i=0}^{4} d_i z^i \tag{5.2. 7}$$

in the inhomogeneous equation and identifying coefficients the following values for the particular integral is obtained

$$\begin{aligned}
d_4 &= \frac{4}{81}\xi_o^6\frac{1}{40}, \\
d_3 &= \frac{4}{81}\xi_o^6(-4+\frac{48}{40})\frac{1}{19} \\
d_2 &= \frac{4}{81}\xi_o^6(6+27(-4+\frac{48}{40})\frac{1}{19})\frac{1}{4}, \\
d_1 &= \frac{4}{81}\xi_o^6(4-12(6+27(-4+\frac{48}{40})\frac{1}{19})\frac{1}{4})\frac{1}{4}, \\
d_o &= \frac{4}{81}\xi_o^6(-1-3(4-12(6+27(-4+\frac{48}{40})\frac{1}{19})\frac{1}{4})\frac{1}{4})\frac{1}{8}.
\end{aligned} \tag{5.2. 8}$$

The solution of equation (5.2.3) is thus given by

$$\chi_{22}(z) = \chi_{22h}(z) + \chi_{22i}(z)\ . \tag{5.2. 9}$$

The boundary conditions to be satisfied is

(b3) u_x is finite for $r = 0$ or, expressed in the z-variable, for $z = 0$.

(b4) $u_x = 0$ on the boundary $z = 1$.

As the polynomial part of the solution is finite for $z = 0$ it is easily seen from the Gauss series representation of the hypergeometric function that $C_{h2} = 0$ in order to satisfy (b3).

From (5.2.3) it is seen, that the polynomial solution will be zero for $z = 1$. Thus the boundary condition (b4) imply that $C_{h1} = 0$, as F(.,.,.;1) is different from zero [7, p.282].

The solution is thus given by

$$\chi_{22}(z) = \sum_{i=o}^{4} d_i z^i \ , \tag{5.2. 10}$$

where the coefficients d_i are defined by (5.2.8).

Transforming to the basic variables we finally get for the axial velocity deficit

$$\begin{aligned} u_x &= U_\infty \left\{ \left(c_w F x^{-2}\right)^{\frac{1}{3}} \chi_{11}\left(r(3c_1^2 c_w F x)^{-\frac{1}{3}}\right) \right. \\ &+ \left. \left(c_w F x^{-2}\right)^{\frac{2}{3}} \chi_{22}\left(r^{\frac{3}{2}}(c_w F x)^{-\frac{1}{2}}(\frac{105}{2\pi}c_1^2)^{-\frac{3}{10}}\right)\right\} \ . \end{aligned} \tag{5.2. 11}$$

The radial velocity is obtained directly from formula (B.1) in Appendix B:

$$\begin{aligned} u_r = \frac{U_\infty}{3}(c_w F)^{\frac{2}{3}}\{x^{-\frac{4}{3}}\zeta\chi_1(\zeta) \ &+ \ x^{-2}(c_w F)^{\frac{1}{3}}\zeta\chi_2(\zeta) \\ &+ \ x^{-2}(c_w F)^{\frac{1}{3}}\frac{2}{\zeta}\int_o^\zeta \chi_2(\zeta)\zeta d\zeta\} \ . \end{aligned}$$

The relation between ζ and z, and ζ and ξ is given by (5.2.1) and (4.2.9), respectively. The above can thus be reformulated as

$$\begin{aligned} u_r &= \frac{U_\infty}{3}(c_w F)^{\frac{2}{3}}\left\{x^{-\frac{4}{3}}\zeta\chi_1(\zeta) + x^{-2}z^{\frac{2}{3}}(3c_1^2 c_w F)^{\frac{1}{3}}\xi_o\chi_{22}(z)\right. \\ &+ \left. 2x^{-2}z^{-\frac{2}{3}}(c_w F)^{\frac{1}{3}}(3c_1^2)^{-\frac{1}{3}}\xi_o^{-1}\int_o^z \frac{2}{3}z^{\frac{1}{3}}(3c_1^2)^{\frac{2}{3}}\xi_o^2\chi_{22}(z)dz\right\} \\ &= \frac{U_\infty}{3}(c_w F)^{\frac{2}{3}}\left\{x^{-\frac{4}{3}}\zeta\chi_1(\zeta) + x^{-2}(3c_1^2 c_w F)^{\frac{1}{3}}\xi_o z^{\frac{2}{3}}\chi_{22}(z)\right. \\ &+ \left. \frac{4}{3}x^{-2}(3c_1^2 c_w F)^{\frac{1}{3}}\xi_o z^{-\frac{2}{3}}\int_o^z z^{\frac{1}{3}}\left\{\sum_{i=0}^{4} d_i z^i\right\}dz\right\} \\ &= \frac{U_\infty}{3}(c_w F)^{\frac{2}{3}}\left\{x^{-\frac{4}{3}}(3c_1^2)^{\frac{1}{3}}\xi\chi_{11}(\xi) + x^{-2}(3c_1^2 c_w F)^{\frac{1}{3}}\xi_o z^{\frac{2}{3}}\chi_{22}(z)\right. \\ &+ \left. \frac{4}{3}x^{-2}(3c_1^2 c_w F)^{\frac{1}{3}}\xi_o z^{\frac{2}{3}}P(z)\right\} \ , \end{aligned} \tag{5.2. 12}$$

where $P(z)$ is given by

$$P(z) = \sum_{i=0}^{4} e_i z^i \,, \tag{5.2. 13}$$

and

$$\begin{aligned} e_4 &= \frac{3}{17} d_4 \,, \\ e_3 &= \frac{3}{13} d_3 \,, \\ e_2 &= \frac{3}{10} d_2 \,, \\ e_1 &= \frac{3}{7} d_1 \,, \\ e_0 &= \frac{3}{4} d_0 \,. \end{aligned} \tag{5.2. 14}$$

Transformed to the basic variables we have:

$$\begin{aligned} u_r &= \frac{U_\infty}{3} (c_w F)^{\frac{2}{3}} \left\{ x^{-\frac{5}{3}} r (3c_1^2 c_w F)^{-\frac{1}{3}} \chi_{11} \left(r(3c_1^2 c_w F x)^{-\frac{1}{3}} \right) \right. \\ &+ x^{-\frac{7}{3}} r \chi_{22} \left(r^{\frac{3}{2}} (c_w F x)^{-\frac{1}{2}} \left(\frac{105}{2\pi} c_1^2 \right)^{-\frac{3}{10}} \right) \\ &+ \left. \frac{4}{3} x^{-\frac{7}{3}} r P \left(r^{\frac{3}{2}} (c_w F x)^{-\frac{1}{2}} \left(\frac{105}{2\pi} c_1^2 \right)^{-\frac{3}{10}} \right) \right\} \,. \end{aligned} \tag{5.2. 15}$$

The boundary is of course still given by the expression obtained in section 4.

Summing up, it is demonstrated that the second order axial velocity field in the wake is obtained from (5.2.11), (5.2.10), (5.2.8) and (4.2.12), and that the second order radial velocity field is given by (5.2.15), (5.2.13-14), (5.2.10), (5.2.8) and (4.2.12).

5.3 Calibration of the model

As with the first order model it is nescessary to determine the unknown constants x_o and c_1 before the second order model can be brought into play.

We adopt the same *assumptions* as in section 4.3 and demand that the wake at the position x_o equals the size of the rotor and that a measurement at the axis down stream the rotor is satisfied. With the nomenclature introduced in section 4.3 we find:

$$D = 2\alpha_1 c_1^{\frac{2}{5}} x_o^{\frac{1}{3}} \tag{5.3. 1}$$

and

$$\begin{aligned} U_m = U_\infty\{1 &- (c_w F(x_o + \Delta x)^{-2})^{\frac{1}{3}} \chi_{11}(0) \\ &- (c_w F(x_o + \Delta x)^{-2})^{\frac{2}{3}} \chi_{22}(0)\} \ , \end{aligned} \tag{5.3. 2}$$

where it has been utilized that the radial velocity component vanish at the axis due to symmetry. This is also seen by a direct calculation of (5.2.12).

From (4.2.12-14) we have immediately

$$\chi_{11}(0)(c_w F)^{\frac{1}{3}}(x_o + \Delta x)^{-\frac{2}{3}} = \beta_1 c_1^{-\frac{4}{5}}(x_o + \Delta x)^{-\frac{2}{3}}. \tag{5.3. 3}$$

From (5.2.10), (5.2.8) and (4.2.14) we have

$$\chi_{22}(0)(c_w F)^{\frac{2}{3}}(x_o + \Delta x)^{-\frac{4}{3}} = \delta_1 c_1^{-\frac{8}{5}}(x_o + \Delta x)^{-\frac{4}{3}}, \tag{5.3. 4}$$

where δ_1 is defined by

$$\begin{aligned}\delta_1 &= (c_w F)^{\frac{2}{3}} \left(\frac{35}{2\pi}\right)^{\frac{6}{5}} 3^{-\frac{4}{5}} \\ &\cdot \frac{1}{162}(-1-3(4-12(6+27(-4+\frac{48}{40})\frac{1}{19})\frac{1}{4})\frac{1}{4}) . \qquad (5.3.\ 5)\end{aligned}$$

Thus we have the following relation

$$U_m = U_\infty\{1-\beta_1 c_1^{-\frac{4}{5}}(x_o+\Delta x)^{-\frac{2}{3}} - \delta_1 c_1^{-\frac{8}{5}}(x_o+\Delta x)^{-\frac{4}{3}}\} ,$$

or

$$\begin{aligned}(U_\infty - U_m)\left\{c^{\frac{4}{5}}(x_o+\Delta x)^{\frac{2}{3}}\right\}^2 &- U_\infty\beta_1\left\{c_1^{\frac{4}{5}}(x_o+\Delta x)^{\frac{2}{3}}\right\} \\ &- U_\infty\delta_1 = 0. \qquad (5.3.\ 6)\end{aligned}$$

The above is a second order equation in $c_1^{\frac{4}{5}}(x_o+\Delta x)^{\frac{2}{3}}$, and the solution can be expressed as

$$c_1^{\frac{4}{5}}(x_o+\Delta x)^{\frac{2}{3}} = \frac{U_\infty\beta_1 \pm (U_\infty^2\beta_1^2 + 4(U_\infty - U_m)U_\infty\delta_1)^{\frac{1}{2}}}{2(U_\infty - U_m)} .$$

Now, from (4.3.4) and (5.3.5) it is seen, that $\beta_1 > 0$ and that $\delta_1 > 0$, as $(c_w F)$ is obviously positive. Of physical reasons it is furthermore clear, that $c_1^{\frac{4}{5}}(x_o+\Delta x)^{\frac{2}{3}} > 0$, and that $(U_\infty - U_m) > 0$. Thus, to get a consistent solution we must choose:

$$c_1^{\frac{4}{5}}(x_o+\Delta x)^{\frac{2}{3}} = \gamma_1 , \qquad (5.3.\ 7)$$

where

$$\gamma_1 \equiv \frac{U_\infty \beta_1 + (U_\infty^2 \beta_1^2 + 4(U_\infty - U_m) U_\infty \delta_1)^{\frac{1}{2}}}{2(U_\infty - U_m)} \ . \tag{5.3. 8}$$

Combining (5.3.1) and (5.3.7) the following expressions are easily obtained:

$$x_o = \left\{ \gamma_1^{\frac{3}{2}} \left(\frac{D}{2\alpha_1} \right)^{-3} - 1 \right\}^{-1} \Delta x \ , \tag{5.3. 9}$$

and

$$c_1 = \left(\frac{D}{2\alpha_1} \right)^{\frac{5}{2}} x_o^{-\frac{5}{6}} . \tag{5.3. 10}$$

Thus adopting the drag coefficient from a direct aerodynamic calculation or from the empirical procedure given in Appendix D, the system consisting of (5.3.9-10), (4.2.12), (5.2.8), (5.2.10-11) and (5.2.13-15) offer a second order approximation of the velocity field in the wake.

6 PC wake program

Each of the models described in section 4 and section 5, respectively, have been implemented in a menu driven interactive program. The packages are designed for use on a PC and written in Turbo-Pascal.

When starting the program you enter the WAKE DEFINITION MENU, which is illustrated in Fig. 6.1 below.

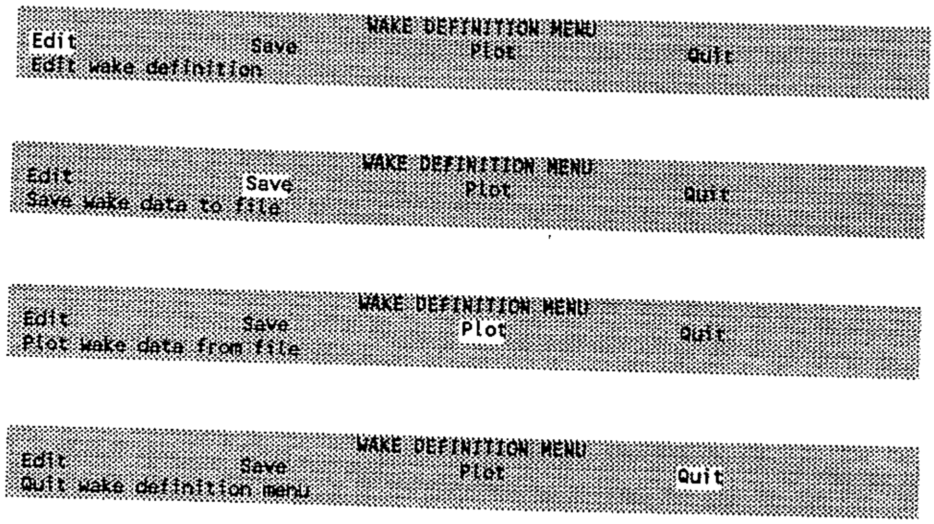


Figure 6.1: WAKE DEFINITION MENU.

As shown this menu offers the following four facilities:

- edit wake definition
- save wake data to file
- plot wake from file
- quit wake definition menu.

By high-lightning a field a short hand explanation of the particular option appears below. Subsequently pressing RETURN the high-lighted option is chosen. The position of the high-lighted field in the WAKE DEFINITION MENU, is controlled by arrows, or by pressing the leading letter in the actual option. From each of the options it is possible to return to the main menu by pressing ESC. If ESC is pressed from the main menu the program quits. The default high-lighted field entering the program is EDIT.

The EDIT option is used to define the actual problem. The WAKE DEFINITION EDITOR is illustrated in Fig. 6.2 below.

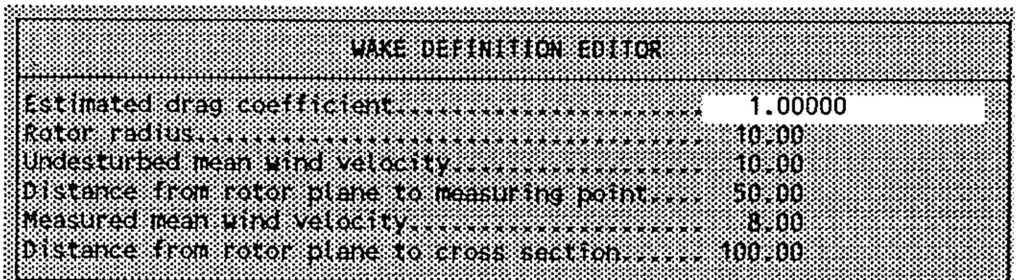


Figure 6.2: WAKE DEF. EDITOR.

As it appears 6 values have to be specified and each of those is described by a text. A particular value can be set by operating a high-lighted field by the arrows, specifying the actual value and finally pressing RETURN.

A check list has been implemented in order only to accept resonable values. If a value is rejected the previous value is retained. The values shown in Fig. 6.2 are the default values.

The SAVE option initiate a calculation of the wake problem currently defined in the WAKE DEFINITION EDITOR. Before starting the calculation you must however specifies a file on which the result can be stored. It is possible to omit the calculation by pressing ESC *before* pressing RETURN after entering the file name.

A check on the file name is performed, and if the specified file name is rejected, the program allows for another attempt.

A succesfull run results in a data file containing an identification, certain key parameters and the wake data. The identification parameter specifies that the present file contain a wake calculation, and is utilized by the PLOT option. The key parameters are the 6 values from the WAKE DEFINITION EDITOR.

The wake data consist of 3 x 51 values describing $(U_\infty + u_x), u_r$ and $((U_\infty + u_x)^2 + u_r^2)^{\frac{1}{2}}$ as a function of the radial coordinate.

The PLOT option offers a graphical presentation of calculated values on a specified file.

The specified file name is checked and if the file name is accepted the content is checked by means of the identification parameter. If the file is rejected, the program allows for another attempt. A succesfull run results in a Turbo-Pascal plot on the screen as illustrated in Fig. 6.3 below.

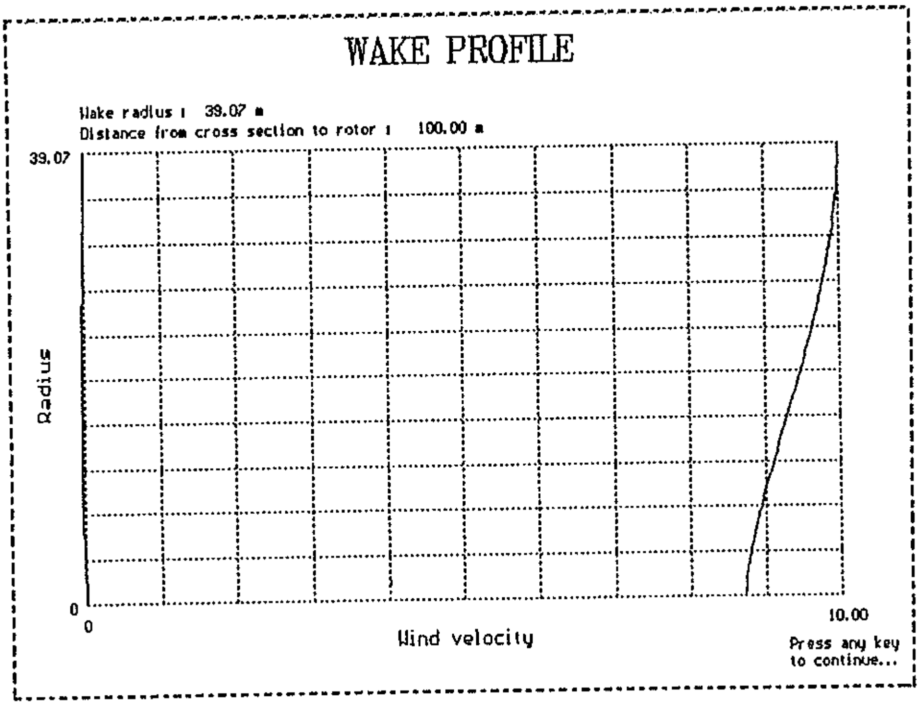


Figure 6.3: Wake plot.

A screen dump will provide the user with a hard copy plot. By pressing any key the program returns to the PLOT option in the main menu.

The QUIT option is used to stop the program. By high-lighting the QUIT option and pressing RETURN the user will leave the program and return to DOS.

7 Conclusions

A first and a second order model for the wake behind a wind turbine, based on the turbulent boundary equations, are established.

The models offer closed form solutions of the axial and radial velocity field as well as the width of the wake at some distance from the turbine. The models do not include the pressure term in the boundary equations, but it is possible to take this into account in a third order model. The models have been made accessible by implementation in a menu driven interactive PC program package written in Turbo-Pascal. The program offers the results on a data file and as a standard plot.

8 References

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Appendix A

First order equation expressed in the ζ-variable.

From (4.2.5) it is seen, that the first order equation can be expressed as

$$U_\infty^2 (c_w F)^{\frac{1}{3}} \frac{\partial}{\partial x}\left[x^{-\frac{2}{3}}\chi_1(\zeta)\right] =$$

$$\frac{1}{r} c_1^2 U_\infty^2 (c_w F)^{\frac{1}{3}} \frac{\partial}{\partial r}\left[c_w F x^{\frac{2}{3}} r \left\{\frac{\partial}{\partial r}\left[x^{-\frac{2}{3}}\chi_1(\zeta)\right]\right\}^2\right] ,$$

or

$$\frac{\partial}{\partial x}\left[x^{-\frac{2}{3}}\chi_1(\zeta)\right] = \frac{1}{r} c_1^2 x^{-\frac{2}{3}} \frac{\partial}{\partial r}\left[c_w F r \left\{\frac{\partial}{\partial r}[\chi_1(\zeta)]\right\}^2\right] .$$

Applying (4.2.2) we obtain

$$\begin{aligned} -\frac{2}{3} x^{-\frac{5}{3}}\chi_1(\zeta) \quad &+ \quad x^{-\frac{2}{3}} \frac{d}{d\zeta}[\chi_1(\zeta)] \frac{\partial \zeta}{\partial x} \\ &= \quad \frac{1}{r} c_1^2 x^{-\frac{2}{3}} \frac{\partial}{\partial r}\left[c_w F r \left\{\frac{d}{d\zeta}[\chi_1(\zeta)] \frac{\partial \zeta}{\partial r}\right\}^2\right] , \end{aligned}$$

or

$$
\begin{aligned}
-\frac{2}{3}x^{-\frac{5}{3}}\chi_1(\zeta) \quad - \quad & \frac{1}{3}x^{-\frac{5}{3}}\zeta\frac{d}{d\zeta}[\chi_1(\zeta)] \\
= \quad & \frac{1}{r}c_1^2 x^{-\frac{2}{3}}\frac{\partial}{\partial r}\left[c_w F r (c_w F x)^{-\frac{2}{3}}\left\{\frac{d}{d\zeta}[\chi_1(\zeta)]\right\}^2\right] \\
= \quad & \frac{1}{r}c_1^2 x^{-1}(c_w F)^{\frac{2}{3}}\frac{\partial}{\partial r}\left[\zeta\left\{\frac{d}{d\zeta}[\chi_1(\zeta)]\right\}^2\right] \\
= \quad & \frac{1}{\zeta}c_1^2 x^{-\frac{4}{3}}(c_w F)^{\frac{1}{3}}\frac{d}{d\zeta}\left[\zeta\left\{\frac{d}{d\zeta}[\chi_1(\zeta)]\right\}^2\right]\frac{\partial\zeta}{\partial r} \\
= \quad & \frac{1}{\zeta}c_1^2 x^{-\frac{5}{3}}\frac{d}{d\zeta}\left[\zeta\left\{\frac{d}{d\zeta}[\chi_1(\zeta)]\right\}^2\right],
\end{aligned}
$$

and thereby

$$
-\frac{2}{3}\zeta\chi_1(\zeta) - \frac{1}{3}\zeta^2\frac{d}{d\zeta}[\chi_1(\zeta)] = c_1^2\frac{d}{d\zeta}\left[\zeta\left\{\frac{d}{d\zeta}[\chi_1(\zeta)]\right\}^2\right]. \tag{A.1}
$$

Appendix B

Second order equation expressed in terms of ζ, $\chi_1(\zeta)$ and $\chi_2(\zeta)$.

The second order equation expressed in terms of $\zeta, \chi_1(\zeta)$ and $\chi_2(\zeta)$ is basicly obtained by substituting (5.1.4 - 5) into the equation of motion (3.4), utilizing that the first order solution satisfies the simplified equation of motion(4.1.17) and retaining terms of order $x^{-\frac{7}{3}}$. However, the full equation (3.4) contain the u_r component, and as a first step this has to be eliminated by use of the equation of continuity (3.3). We find:

$$\begin{aligned}
u_r &= -\frac{1}{r}\int_o^r \frac{\partial}{\partial x}(u_x r)dr \\
&= -\frac{1}{r}(c_w F x)^{\frac{2}{3}}\int_o^r \frac{\partial}{\partial x}(u_x r)(c_w F x)^{-\frac{2}{3}}dr \\
&= -\frac{1}{\zeta}(c_w F x)^{\frac{1}{3}}U_\infty \int_o^\zeta \frac{\partial}{\partial x}\left[(c_w F x^{-2})^{\frac{1}{3}}\chi_1(\zeta) + (c_w F x^{-2})^{\frac{2}{3}}\chi_2(\zeta)\right] r(c_w F x)^{-\frac{1}{3}}d\zeta \\
&= -\frac{1}{\zeta}(c_w F x)^{\frac{1}{3}}U_\infty \int_o^\zeta \frac{\partial}{\partial x}\left[(c_w F x^{-2})^{\frac{1}{3}}\chi_1(\zeta) + (c_w F x^{-2})^{\frac{2}{3}}\chi_2(\zeta)\right] \zeta d\zeta \ .
\end{aligned}$$

Substituting (4.2.17) into the above equation yields

$$\begin{aligned}
u_r &= \frac{U_\infty}{3}(c_w F x^{-2})^{\frac{2}{3}}\zeta\chi_1(\zeta) \\
&- \frac{1}{\zeta}(c_w F)^{\frac{3}{3}}x^{\frac{1}{3}}U_\infty \int_o^\zeta \left(-\frac{4}{3}x^{-\frac{7}{3}}\chi_2(\zeta) + x^{-\frac{4}{3}}\frac{d}{d\zeta}[\chi_2(\zeta)]\frac{\partial\zeta}{\partial x}\right)\zeta d\zeta
\end{aligned}$$

$$
\begin{aligned}
&= \frac{U_\infty}{3}(c_w F x^{-2})^{\frac{2}{3}} \zeta \chi_1(\zeta) \\
&- \frac{1}{\zeta}(c_w F)^{\frac{2}{3}} x^{\frac{1}{3}} U_\infty \int_o^\zeta \left(-\frac{4}{3} x^{-\frac{7}{3}} \chi_2(\zeta) - \frac{1}{3} r (c_w F x)^{-\frac{1}{3}} x^{-\frac{7}{3}} \frac{d}{d\zeta}[\chi_2(\zeta)] \right) \zeta d\zeta \\
&= \frac{U_\infty}{3}(c_w F x^{-2})^{\frac{2}{3}} \zeta \chi_1(\zeta) \\
&+ \frac{U_\infty}{3}(c_w F)^{\frac{3}{3}} x^{-2} \frac{1}{\zeta} \int_o^\zeta \left(4\chi_2(\zeta)\zeta + \frac{d}{d\zeta}[\chi_2(\zeta)]\zeta^2 \right) d\zeta \\
&= \frac{U_\infty}{3}(c_w F x^{-2})^{\frac{2}{3}} \zeta \chi_1(\zeta) \\
&+ \frac{U_\infty}{3}(c_w F)^{\frac{3}{3}} x^{-2} \frac{1}{\zeta} \int_o^\zeta \left(2\chi_2(\zeta)\zeta + \frac{d}{d\zeta}[\chi_2(\zeta)]\zeta^2 \right) d\zeta \\
&+ \frac{U_\infty}{3}(c_w F)^{\frac{3}{3}} x^{-2} \frac{2}{\zeta} \int_o^\zeta \chi_2(\zeta)\zeta d\zeta \\
&= \frac{U_\infty}{3}(c_w F)^{\frac{2}{3}} \left\{ x^{-\frac{4}{3}} \zeta \chi_1(\zeta) + x^{-2}(c_w F)^{\frac{1}{3}} \zeta \chi_2(\zeta) \right. \\
&+ \left. x^{-2}(c_w F)^{\frac{1}{3}} \frac{2}{\zeta} \int_o^\zeta \chi_2(\zeta)\zeta d\zeta \right\} . \qquad \text{(B.1)}
\end{aligned}
$$

We are now ready to set up the resulting equation for the second order problem. By evaluating the order of magnitude of the various terms in (3.4), as in section (4.1), but with corrected expressions for u_x and u_r, and only retaining terms of order $x^{-\frac{7}{3}}$ we get the following contributions:

$$
\begin{aligned}
U_\infty \frac{\partial u_x}{\partial x} &= U_\infty^2 \left\{ -\frac{2}{3}(c_w F)^{\frac{1}{3}} x^{-\frac{5}{3}} \chi_1(\zeta) - \frac{1}{3}(c_w F)^{-\frac{1}{3}} x^{-\frac{5}{3}} \zeta \frac{d}{d\zeta}[\chi_1(\zeta)] \right. \\
&- \left. \frac{4}{3}(c_w F)^{\frac{2}{3}} x^{-\frac{7}{3}} \chi_2(\zeta) - \frac{1}{3}(c_w F)^{\frac{2}{3}} x^{-\frac{7}{3}} \zeta \frac{d}{d\zeta}[\chi_1(\zeta)] \right\}, \qquad \text{(B.2)}
\end{aligned}
$$

$$
\begin{aligned}
u_x \frac{\partial u_x}{\partial x} &= U_\infty^2 \left\{(c_w F)^{\frac{1}{3}} x^{-\frac{2}{3}} \chi_1(\zeta) + (c_w F)^{\frac{2}{3}} x^{-\frac{4}{3}} \chi_2(\zeta)\right\} \\
&\cdot \left\{-\frac{2}{3}(c_w F)^{\frac{1}{3}} x^{-\frac{5}{3}} \chi_1(\zeta) - \frac{1}{3}(c_w F)^{\frac{1}{3}} x^{-\frac{5}{3}} \zeta \frac{d}{d\zeta}[\chi_1(\zeta)]\right. \\
&- \left.\frac{4}{3}(c_w F)^{\frac{2}{3}} x^{-\frac{7}{3}} \chi_2(\zeta) - \frac{1}{3}(c_w F)^{\frac{2}{3}} x^{-\frac{7}{3}} \zeta \frac{d}{d\zeta}[\chi_2(\zeta)]\right\} \\
&\simeq U_\infty^2 \left\{-\frac{2}{3}(c_w F)^{\frac{2}{3}} x^{-\frac{7}{3}} \chi_1(\zeta)^2 - \frac{1}{3}(c_w F)^{\frac{2}{3}} x^{-\frac{7}{3}} \zeta \chi_1(\zeta) \frac{d}{d\zeta}[\chi_1(\zeta)]\right\} \\
&= -\frac{U_\infty^2}{3} \left\{2(c_w F)^{\frac{2}{3}} x^{-\frac{7}{3}} \chi_1(\zeta)^2 + (c_w F)^{\frac{2}{3}} x^{-\frac{7}{3}} \zeta \chi_1(\zeta) \frac{d}{d\zeta}[\chi_1(\zeta)]\right\}, \qquad \text{(B.3)} \\
u_r \frac{\partial u_x}{\partial r} &= \frac{U_\infty}{3}(c_w F)^{\frac{2}{3}} \left\{x^{-\frac{4}{3}} \zeta \chi_1(\zeta) + x^{-2}(c_w F)^{\frac{1}{3}} \zeta \chi_2(\zeta)\right. \\
&+ \left. x^{-2}(c_w F)^{\frac{1}{3}} \frac{2}{\zeta} \int_o^\zeta \chi_2(\zeta) \zeta d\zeta\right\} U_\infty \left\{x^{-\frac{3}{3}} \frac{d}{d\zeta}[\chi_1(\zeta)] + (c_w F)^{\frac{1}{3}} x^{-\frac{5}{3}} \frac{d}{d\zeta}[\chi_2(\zeta)]\right\} \\
&\simeq \frac{U_\infty^2}{3}(c_w F)^{\frac{2}{3}} x^{-\frac{7}{3}} \zeta \chi_1(\zeta) \frac{d}{d\zeta}[\chi_1(\zeta)], \qquad \text{(B.4)}
\end{aligned}
$$

and finally, by use of (4.2.4)

$$
\begin{aligned}
&\frac{1}{r}\frac{\partial}{\partial r}\left[l^2 r \left(\frac{\partial u_x}{\partial r}\right)^2\right] \\
&= \frac{1}{r} l^2 \left(\frac{\partial u_x}{\partial r}\right)^2 + l^2 2 \frac{\partial u_x}{\partial r} \frac{\partial^2 u_x}{\partial r^2} \\
&= \frac{1}{\zeta}(c_w F x)^{-\frac{1}{3}} c_1^2 (c_w F x)^{\frac{2}{3}} U_\infty^2 \left\{x^{-\frac{3}{3}} \frac{d}{d\zeta}[\chi_1(\zeta)] + (c_w F)^{\frac{1}{3}} x^{-\frac{5}{3}} \frac{d}{d\zeta}[\chi_2(\zeta)]\right\}^2 \\
&+ 2 c_1^2 (c_w F x)^{\frac{2}{3}} U_\infty^2 \left\{x^{-\frac{3}{3}} \frac{d}{d\zeta}[\chi_1(\zeta)] + (c_w F)^{\frac{1}{3}} x^{-\frac{5}{3}} \frac{d}{d\zeta}[\chi_2(\zeta)]\right\} \\
&\cdot \left\{(c_w F)^{-\frac{1}{3}} x^{-\frac{4}{3}} \frac{d^2}{d\zeta^2}[\chi_1(\zeta)] + x^{-\frac{6}{3}} \frac{d^2}{d\zeta^2}[\chi_2(\zeta)]\right\}
\end{aligned}
$$

$$
\begin{aligned}
= \;& \frac{1}{\zeta}(c_w F x)^{\frac{1}{3}} c_1^2 U_\infty^2 \left\{ x^{-\frac{3}{3}} \frac{d}{d\zeta}[\chi_1(\zeta)] + (c_w F)^{\frac{1}{3}} x^{-\frac{5}{3}} \frac{d}{d\zeta}[\chi_2(\zeta)] \right\}^2 \\
+ \;& 2c_1^2 (c_w F x)^{\frac{2}{3}} U_\infty^2 \left\{ x^{-\frac{3}{3}} \frac{d}{d\zeta}[\chi_1(\zeta)] + (c_w F)^{\frac{1}{3}} x^{-\frac{5}{3}} \frac{d}{d\zeta}[\chi_2(\zeta)] \right\} \\
\cdot \;& \left\{ (c_w F)^{-\frac{1}{3}} x^{-\frac{4}{3}} \frac{d^2}{d\zeta^2}[\chi_1(\zeta)] + x^{-\frac{6}{3}} \frac{d^2}{d\zeta^2}[\chi_2(\zeta)] \right\} \\
\simeq \;& \frac{1}{\zeta}(c_w F)^{\frac{1}{3}} c_1^2 U_\infty^2 x^{-\frac{5}{3}} \left\{ \frac{d}{d\zeta}[\chi_1(\zeta)] \right\}^2 \\
+ \;& \frac{2}{\zeta}(c_w F)^{\frac{2}{3}} c_1^2 U_\infty^2 x^{-\frac{7}{3}} \frac{d}{d\zeta}[\chi_1(\zeta)] \frac{d}{d\zeta}[\chi_2(\zeta)] \\
+ \;& 2c_1^2 (c_w F x)^{\frac{2}{3}} U_\infty^2 \left\{ (c_w F)^{-\frac{1}{3}} x^{-\frac{7}{3}} \frac{d}{d\zeta}[\chi_1(\zeta)] \frac{d^2}{d\zeta^2}[\chi_1(\zeta)] \right. \\
+ \;& \left. x^{-\frac{9}{3}} \frac{d}{d\zeta}[\chi_1(\zeta)] \frac{d^2}{d\zeta^2}[\chi_2(\zeta)] + x^{-\frac{9}{3}} \frac{d^2}{d\zeta^2}[\chi_1(\zeta)] \frac{d}{d\zeta}[\chi_2(\zeta)] \right\} .
\end{aligned}
\tag{B.5}
$$

Thus substituting (B.2-5) into (3.4) and reducing (all the terms behaving like $x^{-\frac{5}{3}}$ satisfy the first order equation and can thereby be removed), we obtain:

$$
\begin{aligned}
- \;& \frac{4}{3}\chi_2(\zeta) - \frac{1}{3}\zeta \frac{d}{d\zeta}[\chi_2(\zeta)] - \frac{2}{3}\chi_1(\zeta)^2 - \frac{1}{3}\chi_1(\zeta)\zeta \frac{d}{d\zeta}[\chi_1(\zeta)] \\
+ \;& \frac{1}{3}\zeta\chi_1(\zeta) \frac{d}{d\zeta}[\chi_1(\zeta)] = \frac{2}{\zeta} c_1^2 \frac{d}{d\zeta}[\chi_1(\zeta)] \frac{d}{d\zeta}[\chi_2(\zeta)] \\
+ \;& 2c_1^2 \frac{d}{d\zeta}[\chi_1(\zeta)] \frac{d^2}{d\zeta^2}[\chi_2(\zeta)] + 2c_1^2 \frac{d^2}{d\zeta^2}[\chi_1(\zeta)] \frac{d}{d\zeta}[\chi_2(\zeta)] \;,
\end{aligned}
$$

or

$$
\begin{aligned}
& 4\chi_2(\zeta) + \frac{d}{d\zeta}[\chi_2(\zeta)] \left\{ \zeta + 6c_1^2 \left(\frac{1}{\zeta} \frac{d}{d\zeta}[\chi_1(\zeta)] + \frac{d^2}{d\zeta^2}[\chi_1(\zeta)] \right) \right\} \\
& + 6c_1^2 \frac{d}{d\zeta}[\chi_1(\zeta)] \frac{d^2}{d\zeta^2}[\chi_2(\zeta)] = -2\chi_1(\zeta)^2 .
\end{aligned}
\tag{B.6}
$$

Appendix C

Second order equation expressed in terms of z and $\chi_{22}(z)$.

The second order equation is given by (5.1.15).

Introducing in this equation the relations (5.2.1) and (5.2.2), we obtain:

$$4\chi_{22}(z) + \frac{d}{dz}[\chi_{22}(z)]\frac{dz}{d\zeta}\left\{z^{\frac{2}{3}}(3c_1^2)^{\frac{1}{3}}\xi_o + \right.$$

$$\left. 6c_1^2\left(z^{-\frac{2}{3}}(3c_1^2)^{-\frac{1}{3}}\xi_o^{-1}\frac{d}{d\zeta}[\chi_1(\zeta)] + \frac{d^2}{d\zeta^2}[\chi_1(\zeta)]\right)\right\}$$

$$+6c_1^2\frac{d}{d\zeta}[\chi_1(\zeta)]\frac{d^2}{dz^2}[\chi_{22}(z)]\left(\frac{dz}{d\zeta}\right)^2 = -2\chi_1(\zeta)^2. \qquad \text{(C.1)}$$

For the terms containing $\chi_1(\zeta)$ we find by combining (4.2.12-13) and (4.2.10):

$$\begin{aligned}\frac{d}{d\zeta}[\chi_1(\zeta)] &= \frac{d}{d\xi}[\chi_{11}(\xi)]\frac{d\xi}{d\zeta} \\ &= \frac{1}{3}\left[-\xi^2 + \xi_o^{\frac{2}{3}}\xi^{\frac{1}{2}}\right](3c_1^2)^{-\frac{1}{3}}\ ,\end{aligned} \qquad \text{(C.2)}$$

$$
\begin{aligned}
\frac{d^2}{d\zeta^2}[\chi_1(\zeta)] &= \frac{d}{d\zeta}\left[\frac{d}{d\zeta}[\chi_1(\zeta)]\right] \\
&= \frac{1}{3}\frac{d}{d\xi}[-\xi^2 + \xi_o^{\frac{3}{2}}\xi^{\frac{1}{2}}](3c_1^2)^{-\frac{2}{3}} \\
&= [-\frac{2}{3}\xi + \frac{1}{6}\xi_o^{\frac{3}{2}}\xi^{-\frac{1}{2}}](3c_1^2)^{-\frac{2}{3}} \ ,
\end{aligned}
\tag{C.3}
$$

and

$$
\begin{aligned}
\chi_1(\zeta) &= \chi_{11}(\xi) \\
&= -\left(\frac{1}{3}\xi^{\frac{3}{2}} - \frac{1}{3}\xi_o^{\frac{3}{2}}\right)^2 .
\end{aligned}
\tag{C.4}
$$

Combining (C.1-4) we find:

$$
\begin{aligned}
4\chi_{22}(z) &+ \frac{d}{dz}[\chi_{22}(z)]\frac{3}{2}\zeta^{\frac{1}{2}}(3c_1^2)^{-\frac{1}{2}}\xi_o^{-\frac{3}{2}}\left\{z^{\frac{2}{3}}(3c_1^2)^{\frac{1}{3}}\xi_o\right. \\
&+ 6c_1^2\left[z^{-\frac{2}{3}}(3c_1^2)^{-\frac{1}{3}}\xi_o^{-1}\frac{1}{3}(-\xi^2 + \xi_o^{\frac{3}{2}}\xi^{\frac{1}{2}})(3c_1^2)^{-\frac{1}{3}}\right. \\
&+ \left.\left.\left(-\frac{2}{3}\xi + \frac{1}{6}\xi_o^{\frac{3}{2}}\xi^{-\frac{1}{2}}\right)(3c_1^2)^{-\frac{2}{3}}\right]\right\} \\
&+ 6c_1^2\frac{1}{3}[-\xi^2 + \xi_o^{\frac{3}{2}}\xi^{\frac{1}{2}}](3c_1^2)^{-\frac{1}{3}}\frac{d^2}{dz^2}[\chi_{22}(z)] \\
&\cdot \frac{9}{4}\zeta(3c_1^2)^{-1}\xi_o^{-3} = -2\left(\frac{1}{3}\xi^{\frac{3}{2}} - \frac{1}{3}\xi_o^{\frac{3}{2}}\right)^4 .
\end{aligned}
\tag{C.5}
$$

The definitions of ζ, ξ and z imply, that

$$
z \equiv \frac{\xi^{\frac{3}{2}}}{\xi_o^{\frac{3}{2}}},
\tag{C.6}
$$

and (C.5) can thus be reduced to

$$
\begin{aligned}
4\chi_{22}(z) &+ \frac{3}{2}\frac{d}{dz}[\chi_{22}(z)]z^{\frac{1}{3}}\xi_o^{-1}\left\{z^{\frac{2}{3}}\xi_o\right. \\
&+ 2\left[z^{-\frac{2}{3}}\xi_o^{-1}\frac{1}{3}\left(-z^{\frac{4}{3}}\xi_o^2 + \xi_o^{\frac{3}{2}}z^{\frac{1}{3}}\xi_o^{\frac{1}{2}}\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \left(-\frac{2}{3}z^{\frac{2}{3}}\xi_o + \frac{1}{6}\xi_o^{\frac{3}{2}}z^{-\frac{1}{3}}\xi_o^{-\frac{1}{2}}\right)\Big]\Big\} \\
&+ \quad \frac{3}{2}\left[-z^{\frac{4}{3}}\xi_o^2 + \xi_o^{\frac{3}{2}}z^{\frac{1}{3}}\xi_o^{\frac{1}{2}}\right]\frac{d^2}{dz^2}[\chi_{22}(z)]z^{\frac{2}{3}}\xi_o^{-2} \\
&= \quad -2\left(\frac{1}{3}z\xi_o^{\frac{3}{2}} - \frac{1}{3}\xi_o^{\frac{3}{2}}\right)^4 ,
\end{aligned}
$$

or

$$
3z(z-1)\frac{d^2}{dz^2}[\chi_{22}(z)] + 3(z-1)\frac{d}{dz}[\chi_{22}(z)] - 8z
$$

$$
= \frac{4}{81}\xi_o^6(z-1)^4 . \tag{C.7}
$$

Appendix D

Simplified estimation of the rotor drag coefficient.

If the turbine investigated in [8] can be considered representative for three bladed turbines in general, as far as the drag coefficient is concerned, the following offers a simpel estimate of the drag coefficient, solely based on knowledge to the undisturbed free mean wind velocity at hub height. The Fig. D.1 below, shows the measured results from [8].

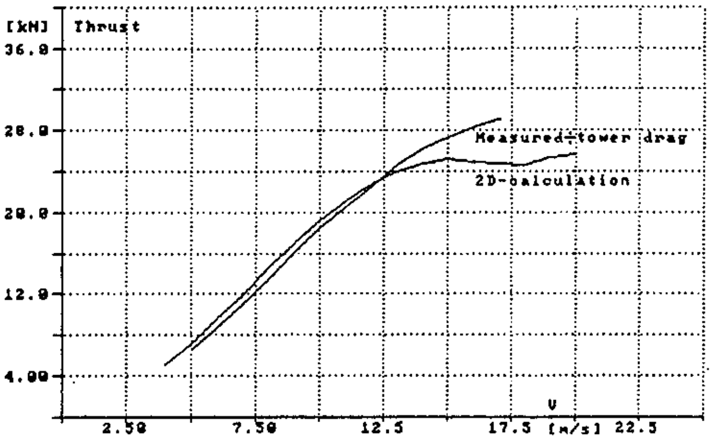

Figure D.1: Axial force as function of the wind velocity for the DANWIN turbine.

The following form of the relation connecting the axial force and the wind velocity is *assumed*

$$F_x = aU^2 + bU + c, \tag{D.1}$$

where U denote the mean wind velocity at hub heigh in m/s, and F_x is the axial force in N. From the curve on Fig. D.1 we get the following values:

$$\begin{aligned}
(F_x, U) &= (8000, 5.73), \\
(F_x, U) &= (18530, 10), \\
(F_x, U) &= (27200, 15) \,.
\end{aligned}$$

Inserting each of those in (D.1) we obtain a system of 3 linear equations in a, b and c. The solution of the system is:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -78.975 \\ 3708.400 \\ -10656.200 \end{pmatrix}. \tag{D.2}$$

Combining (D.1) and (D.2) we obtain

$$F_x = -78.975U^2 + 3708.4U - 10656.2. \tag{D.3}$$

For a body with the drag coefficient c_w the following relation holds:

$$F_x = \frac{1}{2}\rho c_w A U^2, \tag{D.4}$$

where ρ is the density of the air and A is the area of the body seen by the wind. By introducing the values for the rotor examined in [8] and combining (D.3) and (D.4) we obtain

$$c_w = -44.95\frac{1}{U^2} + 15.64\frac{1}{U} - 0.333\ . \tag{D.5}$$

The above relation is considered to be valid within the wind velocity interval in which the formula is calibrated, i.e. for velocities in the interval $5\ \mathrm{m/s} < U < 17.5\ \mathrm{m/s}$.

Bibliographic Data Sheet

Risø National Laboratory | Risø-M-2760

Title and author(s)
A simpel Wake Calculation Procedure

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Date
December 1988

Department or group
Meteorology and Wind Energy

Groups own registration number(s)

Project/Contract number
Nibe-Wake 102-46

Pages
58

Tables

Illustrations
6

References
8

ISBN
87-550-1484-4

Abstract (Max. 2000 char.)
This report deals with a first and a second order model for the description of the wake behind a wind turbine.
The models are based on the turbulent boundary equations, and closed form solutions are obtained for the width of the wake as well as for the mean wind profile in the wake, by adopting a similarity assumption and by utilizing the mixing length theory in the description of the turbulent stresses.
The models have been made accessible by being implemented into an interactive program package designed for use on a personal computer. The use of the package is described.

Descriptors – INIS/EDB
ANALYTICAL SOLUTION; COMPUTER CODES; MATHEMATICAL MODELS; TURBULENCE; VELOCITY; VISCOUS FLOW; WIND TURBINES; WIND TURBINE ARRAYS

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